Topological Entropy of Circle Endomorphisms (力学系における非線形回路の諸問題)

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TOPOLOGICAL ENTROPY OF CIRCLE ENDOMORPHISMS

by

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1° The purpose of this paper is to investigate the topological entropy $h(f)$ of a piecewise monotone and continuous transformation f of the unit circle S^1 and to relate it to the mapping degree $\deg(f)$ of f. For such a transformation f, $L(f)$ denotes the number of the maximal intervals where f is monotone. Then the limit $G(f) = \lim \frac{1}{n} \log L(f^n)$ exists and is called the growth number of f. Our main results are the following theorems 1 and 2.

THEOREM 1 Let f be a piecewise monotone and continuous transformation of S^1. Suppose that f is not a local homeomorphism. Then $h(f) = G(f)$.

THEOREM 2 (1) For a piecewise monotone and continuous transformation f, $h(f) \geqq \log \deg(f)$.

(2) For a local homeomorphism f of S^1, $h(f) = \log \deg(f)$.

REMARK Theorem 2 (1) is the simplest case of a general theorem of Manning ([4]).

In the case of transformations of the interval, the concept of growth number is introduced and extensively studied by Milnor and Thurston ([3]).

Now let us recall the definition of the topological entropy of a continuous transformation of a compact space X ([1]).

$$h(f) = \sup \left\{ H(f, \mathfrak{A}) \,\middle|\, \mathfrak{A} \text{ is a open covering of } X \right\}$$

$$H(f, \mathfrak{A}) = \lim_{n\to\infty} \frac{1}{n} \log N(\mathfrak{A} \vee f^{-1}\mathfrak{A} \vee \ldots\ldots \vee f^{-n+1}\mathfrak{A})$$

Here for open coverings $\mathfrak{A}$, $\mathfrak{B}$ of X, $\mathfrak{A} \vee \mathfrak{B}$ denotes the open covering $\{ A \cap B \mid A \in \mathfrak{A}, B \in \mathfrak{B} \}$, and $N(\mathfrak{A})$ is the smallest cardinality of subcoverings of $\mathfrak{A}$.

2° A nonempty finite subset Δ of the unit circle S^1 is called a <u>partition</u>; its cardinality is denoted by $m(\Delta)$; the closure of a connected component of $S^1 - \Delta$ is called a <u>small interval</u> of Δ; for a piecewise monotone and continuous transformation f, Δ_n denotes the partition $\Delta \cup f^{-1}\Delta \cup f^{-2}\Delta \cup \ldots\ldots \cup f^{-n+1}\Delta$.
A continuous transformation f is called <u>piecewise monotone</u> if there exists a partition Δ such that f is strictly monotone on each small interval of Δ. Such a partition Δ is called a <u>monotone partition</u> for f. Henceforth in this paper, any transformation of S^1 is to be piecewise monotone and continuous.

<u>LEMMA</u> 3 <u>For a monotone partition</u> Δ <u>for</u> f,

$$h(f) \leqq \liminf_{n\to\infty} \frac{1}{n} \log m(\Delta_n)$$

<u>PROOF</u> Given an arbitrary covering $\mathfrak{A}$ of S^1, one can choose a partition Δ' such that (1) Δ' contains as a subset the prescribed monotone partition Δ for f, (2) each small interval

of Δ' is contained in some member of $\mathfrak{a}$. Then $m(\Delta_n') \geqq N(\mathfrak{a} \vee f^{-1}\mathfrak{a} \vee f^{-2}\mathfrak{a} \vee \ldots\ldots \vee f^{-n+1}\mathfrak{a})$. (Each small interval of Δ_n' is contained in at least one member of $\mathfrak{a} \vee f^{-1}\mathfrak{a} \vee \ldots\ldots \vee f^{-n+1}\mathfrak{a}$.) Let J be a small interval of Δ_n. Then because Δ is a monotone partition for f, f^i maps J homeomorphically into some small interval of Δ for each $i \in \{0, 1, \ldots\ldots, n-1\}$. Thus $m(\Delta_n' \cap \mathring{J}) \leqq nb$, where b is so chosen that $b \geqq m(\Delta' \cap \mathring{I})$ for each small interval I of Δ. Hence $m(\Delta_n') \leqq (nb+1)\, m(\Delta_n)$. Thus

$$N(\mathfrak{a} \vee f^{-1}\mathfrak{a} \vee \ldots\ldots \vee f^{-n+1}\mathfrak{a}) \leqq (nb+1)\, m(\Delta_n)$$

Letting $n \to \infty$, we have

$$H(f,\mathfrak{a}) \leqq \liminf_{n \to \infty} \frac{1}{n} \log m(\Delta_n) .$$

Now as $\mathfrak{a}$ was an arbitrary covering of S^1, we have done the proof.

3° A partition Δ is called a fine partition for a transformation f of S^1, in case (1) f embeds each small interval of Δ into S^1, (2) the length of each small interval of Δ, as well as that of its image by f, is smaller than $\frac{1}{3}$ of the whole length of S^1. Thus a fine partition for f is automatically a monotone partion for f. The next lemma is a converse of lemma 3.

LEMMA 4 <u>For a fine partition</u> Δ <u>for</u> f,

$$h(f) \geqq \limsup_{n \to \infty} \frac{1}{n} \log m(\Delta_n) .$$

PROOF Our proof is analogous to the argument employed by Bowen in his paper [2] .

Given a positive integer N, we shall construct an open covering, say $\mathfrak{A}_N$, out of the partition $\Delta = \{x_1, x_2, \ldots, x_r\}$. For each point x_j of Δ , consider the open interval U_j whose endpoints are those points of Δ_N which are next to x_j . These U_j 's, together with interiors of small intervals $\mathring{I}_j$'s of Δ ($1 \leqq j \leqq r$) , constitute an open covering $\mathfrak{A}_N$ of S^1. For each member $B = A_0 \cap f^{-1}A_1 \cap \ldots \cap f^{-n+1}A_{n-1}$ of $\mathfrak{A}_N \vee f^{-1}\mathfrak{A}_N \vee \ldots \vee f^{-n+1}\mathfrak{A}_N$ (A_i is either some U_j or some $\mathring{I}_j$), we shall count the number of the small intervals of Δ_n which intersects B . Notice that a small interval of Δ_n is of the form $I_{i_0} \cap f^{-1}I_{i_1} \cap f^{-2}I_{i_2} \cap \ldots \cap f^{-n+1}I_{i_{n-1}}$. (Here we use the assumption that Δ is fine.) Now look at the sequence $A_0, A_1, \ldots, A_{n-1}$ which defines B . We shall construct a subsequence in the following way. Let i_1 be the smallest number, if any, such that A_{i_1} is equal to some U_j , then delete the (N - 1) terms $A_{i_1+1}, \ldots, A_{i_1+N-1}$ from the sequence. Consider the new sequence, and let i_2 be the next number, if any, such that $A_{i_2} = U_j$ for some U_j . Then delete the N - 1 terms $A_{i_2+1}, \ldots, A_{i_2+N-1}$ from the sequence. Proceeding in this fashion, one obtains finally a subsequence, say, $A_{j_1}, A_{j_2}, \ldots, A_{j_s}$. Notice that if $A_{j_k} = U_j$, Then A_{j_k} is contained in a union of two small intervals of Δ_N. Thus $f^{-j_1}(A_{j_1}) \cap \ldots \cap f^{-j_s}(A_{j_s})$, hence <u>a forteriori</u> B , is contained in a union of at most $2^{\frac{n}{N}+1}$ small intervals of Δ_n . Thus

$$N(\mathfrak{A}_N \vee \ldots \vee f^{-n+1}\mathfrak{A}_N) \cdot 2^{\frac{n}{N}+1} \geq m(\Delta_n) .$$

Letting $n \to \infty$, one gets

$$H(f, \mathfrak{A}_N) + \frac{1}{N}\log 2 \geq \limsup_{n\to\infty} \frac{1}{n}\log m(\Delta_n) .$$

As N is arbitrary, one obtains

$h(f) \geq \limsup_{n \to \infty} \frac{1}{n} \log m(\Delta_n)$, as is desired.

4° In this paragraph we shall complete the proof of theorems 1 and 2. To begin with one has the following easy consequence of lemmas 3 and 4.

PROPOSITION 5 *Let Δ be an arbitrary monotone partition for a transformation f of S^1. Then the following limit exists and is equal to $h(f)$.*

$$\lim_{n\to\infty} \frac{1}{n} \log m(\Delta_n)$$

PROOF Given a monotone partition Δ for f, there exists a fine partition Δ' for f such that $\Delta' \supset \Delta$. Then

$$h(f) \geqq \limsup_{n\to\infty} \frac{1}{n} \log m(\Delta'_n) \geqq \liminf_{n\to\infty} \frac{1}{n} \log m(\Delta'_n)$$
$$\geqq \liminf_{n\to\infty} \frac{1}{n} \log m(\Delta_n) \geqq h(f) .$$

Hence $h(f) = \lim_{n\to\infty} \frac{1}{n} \log m(\Delta_n)$, as is desired.

PROOF OF THEOREM 1 Let f be a transformation of S^1, which is not a local homeomorphism. Let Δ_f be a monotone partition for f with the smallest cardinality. Of course such a partition is unique and each small interval of Δ_f is a maximal interval where f is monotone. Then clearly $L(f) = m((\Delta_f)_n)$. Thus Theorem 1 is implied by Proposition 5.

PROOF OF THEOREM 2 (1) Notice that $m(f^{-1}\{x\}) \geqq \mathrm{deg}(f)$ for each point x of S^1. Thus for a monotone partition Δ for f, one has $m(\Delta_n) \geqq m(f^{-n+1}\Delta) \geqq m(f^{-n+1}\{x_0\}) \geqq (\deg(f))^{n-1}$, where $x_0 \in \Delta$. Hence $h(f) = \lim_n \frac{1}{n} \log m(\Delta_n) \geqq \log \deg(f)$.

PROOF OF THEOREM 2 (2) Let x be an arbitrary point of S^1. Then $\{x\}$ is a monotone partition for a local homeomorphism f. Thus $h(f) = \lim_n \frac{1}{n} \log m(\{x\}_n)$. Now $m(f^{-i}\{x\}) = (\deg(f))^i$. Thus $(\deg(f))^{n-1} \leqq m(\{x\}_n) \leqq 1 + \deg(f) + \deg(f)^2 + \ldots\ldots + \deg(f)^{n-1}$. Hence we get $h(f) = \log \deg(f)$

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